Large time behaviour of the ice cover caused by a load moving along a frozen channel

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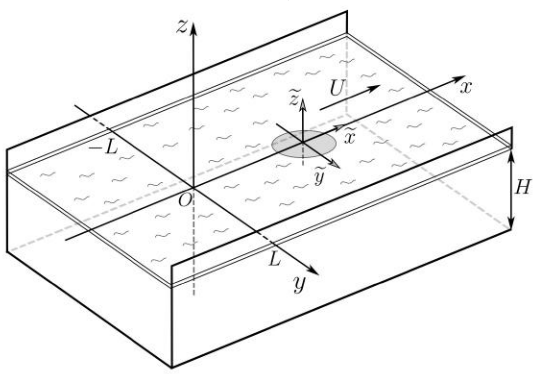
Unsteady response of an ice cover in a channel with vertical walls is studied for large times. The ice deflection is caused by a load moving along the frozen channel at a constant speed. Viscous damping of waves generated by the load and propagating along the channel is not taken into account. Viscous effects are important for the ice deflection. However, viscous damping is not well understood and its modelling is not well established. Models of ice response without damping are less physical, however, they may provide helpful estimates of maximum strains and bearing capacity of the ice cover. Such estimates are known for loads moving on the ice cover of infinite extent. In this paper, estimates of strains and deflections are obtained for a load moving along a frozen channel with vertical side walls.

Introduction and formulation of the problem

The unsteady hydroelastic waves caused by a load moving along a channel covered with ice are considered (see Figure below). The channel has a rectangular cross section with finite depth H, $-H < z < 0$, and width $2L$, $-L < y < L$. The channel is infinitely long, $-\infty < x < \infty$. The channel is filled with incompressible and inviscid liquid of density ρ_l. The flow caused by the ice deflection is potential. The thickness of the ice cover h_i and the ice density ρ_i are constant. The vertical displacement of the ice sheet, $w(x, y, t)$, satisfies the equation of thin elastic plate with clamped boundary conditions at $y = \pm L$. The flow velocity potential $\varphi(x, y, z, t)$ satisfies the Laplace equation, impermeability conditions at the channel walls and at the bottom, and the kinematic and dynamic boundary conditions at the ice-liquid interface. The load is modelled by a localized smooth pressure distribution.



The problems of loads moving along an unbounded ice cover without vertical walls are well studied (see, for example, an excellent review in Squire et al. (1996)). The problems are usually solved within the theory of linear hydroelasticity. It is known that the ice deflection is strongly dependent on the speed of the load. If the load speed is below a certain critical value, the ice deflection is localized near the load and quickly decays with the distance from the load. For higher speeds of the load, outgoing waves are formed in the far field if viscous damping in ice is not included in the mathematical model. At the critical speed, the linear theory without damping predicts unbounded ice response. To obtain the estimates of the ice response for the critical speed of the load either nonlinear effects (Guyenne et al. 2012) or viscous damping (Hosking et al. 1988) or both are included in the ice model. The presence of the channel walls complicates the problem. There are an infinite number of critical speeds for a frozen channel, in contrast to the ice sheet of infinite extent, for which there are two simple critical values of the load speed. Each critical speed for an ice cover in a channel corresponds to a mode of propagating along the channel and sloshing across the channel waves studied by Korobkin et al. (2014). For a speed of the load different from the critical values, the linear theory of hydroelasticity can be used to estimate the strains in the ice cover. We restrict ourselves to a constant and different from the critical values speed of the load and the strain/deflection distributions which are stationary in the coordinate system moving together with the load. These stationary distributions can be obtained directly within ice models with viscous damping, where ice deflection decays exponentially with distance from the moving load, see Shishmarev et al. (2016). Decreasing viscous damping, we obtain higher strains in the ice cover and longer region of significant deflections along the channel. Both deflections and strains are given by infinite series and integrals. Numerical integration becomes challenging for small damping. To estimate the strains, we need them for zero damping, where the approach from Shishmarev et al. (2016) does not work.

Another approach to stationary strain/deflection distributions caused by a load moving along the channel for zero damping in the ice cover (Schulkes et al. 1988) is employed in the present study. In this approach, the stationary distributions are approached in time t starting from the initial rest state. At $t = 0$ the load, which is modelled by an external pressure acting on the ice plate, is at rest. The initial ice deflection satisfies the stationary equation of thin elastic plate with proper boundary conditions on the walls of the channel. The pressure distribution generated by the load is symmetric with respect to the central line of the channel. Then the load starts to move at a constant speed U along the channel. The ice deflection decays far ahead and far behind the moving load as $|x - Ut| \to \infty$ at any finite time t. The unsteady problem of hydroelasticity is solved with the help of the Fourier transform along the channel and the normal-mode method (Khabakhpasheva, 2006) for the ice deflection. A second-order differential equations in time for the principal coordinates of the normal modes are derived and solved analytically. As a result, the ice deflection is presented by an infinite series of regular Fourier integrals. The integrals are evaluated for large times by asymptotic methods, see Schulkes et al. (1988). The asymptotic behaviours of the integrals depend on the speed of the load with respect to the critical speeds of the propagating-sloshing wave modes. It is shown that for large times the ice deflection consists of symmetric deflection localized near the load and a system of waves in front and behind the load. The number of these waves is obtained and the wave amplitudes are evaluated numerically. Each wave propagates along the channel with the speed of the load U. The waves are stationary in the coordinate system moving together with the load.

This study is motivated by experiments in ice tanks, operations on ice in rivers and channels such as cargo transportation or ice breaking to avoid flooding, and ice-structure interaction. The strains calculated with zero damping are higher than the real ones. For safe transportation on ice, one needs to compare the computed strains with a strain critical value and determine safe speed of transportation. Knowing amplitudes of the waves generated by the moving load, we can find places and estimate the values of the highest strains far ahead and behind the load. It is expected that there are such speeds of the load that the maximum strains behind and or in front of the load are achieved at the walls of the channel.

Method of the solution and discussion

The considered initial-boundary value problem is solved with the help of the Fourier transform along the channel. Then the Fourier image of the ice deflection, $w^F(\xi, y, t)$, is sought in the form

$$w^F(\xi, y, t) = \sum_{n=1}^{\infty} a_n(\xi, t)\psi_n(\xi, y), \qquad w^F(\xi, y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(x, y, t) e^{-i\xi x} dx. \tag{1}$$

Here $\psi_n(\xi, y)$ are the modes of the ice cover oscillations in the channel, and $a_n(\xi, t)$ are the principle coordinates of the modes. The functions $\psi_n(k, y)$ were calculated in Korobkin et al. (2014). They describe the hydroelastic waves propagating along the channel

$$w(x, y, t) = \mathrm{Re}\left[A^n \psi_n(k, y) e^{i(kx - \omega_n t)}\right], \tag{2}$$

where $\omega_n(k)$ is the wave frequency, $k > 0$ is the wave number and A^n is the wave amplitude. By the method of separating variables we arrive at the following differential equations for the principle coordinates $a_n(\xi, t)$,

$$\frac{d^2 a_n}{dt^2} + \omega_n^2(\xi) a_n = H_n(\xi) e^{-iU\xi t}, \quad H_n(\xi) = \frac{\int_{-L}^{L} P^F(\xi, y)\psi_n(\xi, y) dy}{\rho_l L \int_{-L}^{L} (\alpha\psi_n(\xi, y) + \varphi_n(\xi, y, 0))\psi_n \, dy},$$

where $P^F(\xi, y)$ is the Fourier transform of the given load distribution, $\alpha = (\rho_i h_i/\rho_l L)$ and $\varphi_n(\xi, y, z)$ is the flow potential corresponding to the mode $\psi_n(\xi, y)$. This equation is solved analytically subject to the initial conditions.

The ice deflection $w(x, y, t)$ is given by the inverse Fourier transform applied to (1), where

$$\int_{-\infty}^{\infty} a_n(\xi,t)\psi_n(\xi,y)e^{i\xi x}d\xi = \int_{-\infty}^{0} a_n(\xi,t)\psi_n(\xi,y)e^{i\xi x}d\xi + \int_{0}^{\infty} a_n(\xi,t)\psi_n(\xi,y)e^{i\xi x}d\xi =$$

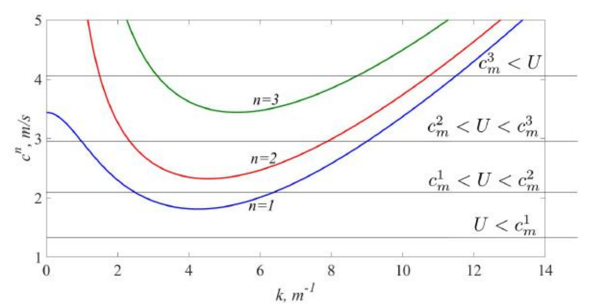
$$= \int_0^{\infty} H_n\psi_n \left(\frac{\xi U}{2\omega_n^2}\left[\frac{e^{i(\xi x+\omega_n t)}}{\omega_n+\xi U} - \frac{e^{i(\xi x-\omega_n t)}}{\omega_n-\xi U} + \frac{e^{-i(\xi x+\omega_n t)}}{\omega_n+\xi U} - \frac{e^{-i(\xi x-\omega_n t)}}{\omega_n-\xi U}\right] + \frac{\cos(\xi(x-Ut))}{2(\omega_n^2-\xi^2U^2)}\right) d\xi. \quad (3)$$

In the numerical calculations, the series (1) for $w^F(\xi, y, t)$ is truncated to N_{mod} terms. We introduce moving coordinate system (X, y, z), $X = x - Ut$, and find asymptotic behaviour of each integral in (3) for each mode number n for large times, $t \to \infty$ and $X = O(1)$. Depending on the load speed U some integrals in (3) have poles at the points where $\omega_n(\xi) = \xi U$. Note that the integral (3) is regular and can be written as the sum of five integrals which are understood as Cauchy principal value integrals, in general. If an integral does not have a pole then it is regular and its contribution is of order $O(1/t)$ as $t \to \infty$. The large-time contributions of non-regular integrals consist of two parts: *(a)* a part which is even in X and decaying as $|X| \to \infty$, *(b)* waves (2) with amplitudes A^n propagating behind and in front of the load. In these waves, $k = \xi_m^n$, $m = 1, 2$, where ξ_m^n are solutions of the equation $\omega_n(\xi) = \xi U$, $\xi_1^n < \xi_2^n$, and the amplitudes A_m^n are given by

$$A_m^n = \frac{2\pi H_n(\xi_m^n)}{\xi_m U(c_g^n(\xi_m^n) - U)},$$

where c_g^n is the group velocity of the n-th mode of hydroelastic wave in the channel. The phase speeds of these waves is equal to the speed of the load, U. These waves do not propagate with respect to the load in the moving coordinate system. The long waves with ξ_1^n are placed behind the load, and the short waves with ξ_2^n are in front of the load.

Calculations were performed for the laboratory ice tank at the Sholem Aleichem Amur State University. The parameters are: $H = 1$m, $L = 1.5$m, $h_i = 3$cm, $\rho_l = 1024\mathrm{kg/m}^3$, $\rho_i = 920\mathrm{kg/m}^3$, Young modulus $E = 4.2 \cdot 10^9\mathrm{N/m}^2$. The size of the area of the load distribution on the ice plate is 15 cm. The calculated phase speeds $c^n(k)$ for $n = 1, 2, 3$ are shown in the figure on the left. Different cases are distinguished depending on the speed of the load. If the speed of the load is less then the minimum of $c^1(k)$ then there are no waves generated by the load. The ice deflection in this case is symmetric in X and localized near the load. The minimum of the phase speed of a wave mode corresponds to a critical speed. At these speeds the phase and group speeds are equal. These resonant cases are not investigated in the present study. For the speed of the load greater than the minimum of $c^n(k)$ for $n = N > 0$ there are points of intersection between the speed of the load and phase curves for the modes with $n \leq N$. For $c^1_{\min} < U < c^2_{\min}$ there are two such points for the first mode, ξ_1^1 and ξ_2^1, where $\xi_1^1 < \xi_m^1 < \xi_2^1$, $c^n(\xi_m^1) = c^n_{\min}$. The ice deflection along the centre line of the channel is shown in Figure 1a by the red dashed line, where the deflection scale, $w_{sc} = P_0/(\rho_l g)$, and P_0 is the magnitude of the load. The black solid line corresponds to the solution of the steady problem with small damping ($\tau = 0.005$ s in Shishmarev et al. (2016)). 3D ice deflections is shown in Figure 1b. One can see the waves behind and in front of the load.



Conclusion. The obtained exact decomposition of the ice deflection to the symmetric, long and short wave parts for large times makes it possible to estimate the maximum strains under the load, far behind and in front of the load. Figure 1c shows that the maximum strains are achieved in the waves in front of the load in this particular case. In general, for supercritical cases ($U > c^1_{\mathrm{min}}$), long waves propagate behind and short waves propagate in front of the load. Phase speeds and, in particular, the value $c^1(0)$ which is finite in contrast to that of other modes, strongly depend on the configuration of the channel and ice thickness. It is possible that $c^1(0) < c^2_{\mathrm{min}}$ and then there is the only one short wave in front of the load for $c^1(0) < U < c^2_{\mathrm{min}}$.
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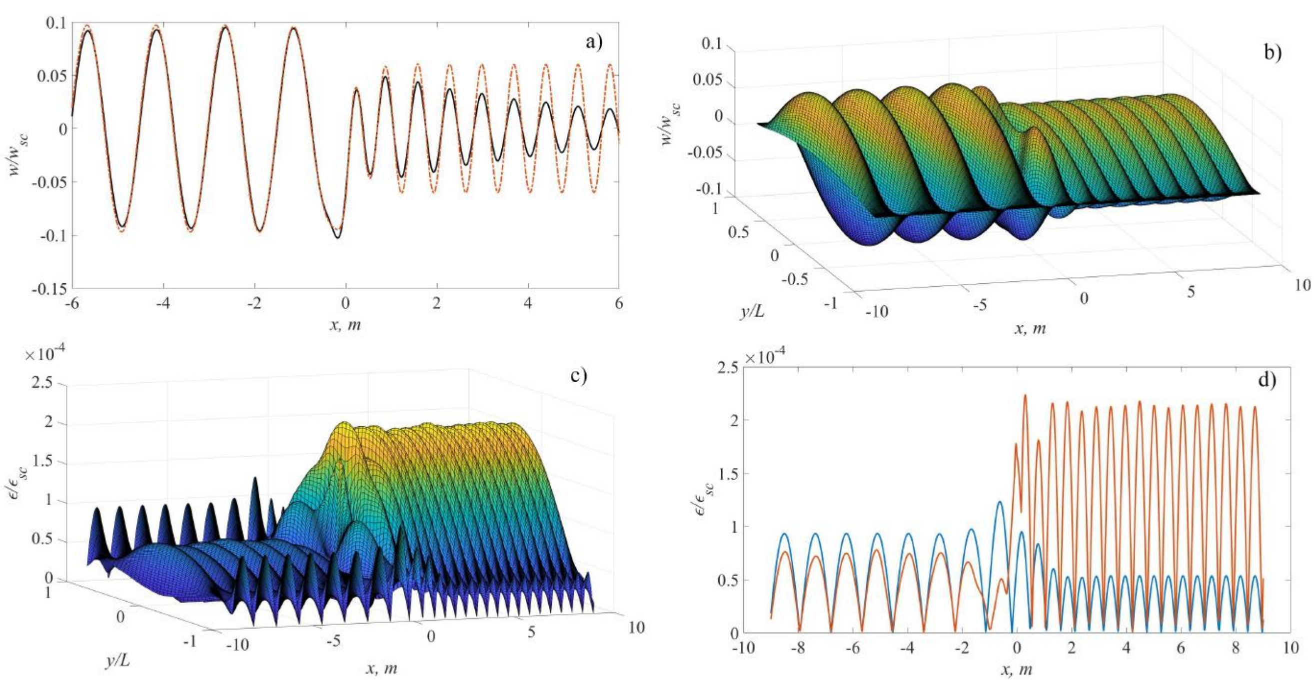

Figure 1: a) Scaled ice deflection along the centre line of the channel for $U = 2$ m/s. Red line is for the asymptotic solution. Black line is for the steady solution with very small damping. b) 3D ice deflection for $U = 2$m/s. c) 3D maximum strain distribution for $U = 2$m/s in the given channel. d) Strains along the centre line of the channel (red line) and at the walls (blue line) for $U = 2$m/s.