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Semi-analytical solution for wave diffraction-radiation by a truncated porous vertical cylinder

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Introduction

The interest for wave/porous body interaction has been recently increased due the expansion of aquaculture and fish farms industry. In addition, structures where porous surface represent an important part (such as floating breakwater [6], Tuned Liquid Damper (TLD) [3],...) are known for their capability to reduce wave loads through energy dissipation. Therefore, it may consists one of the practical solutions to reduce motion response or attenuate waves in some coastal engineering applications. The porosity effect is usually taken into account by introducing a body boundary condition (linear [1] or quadratic [3]) that links pressure to velocity at both sides of the porous boundary.

The main purpose of the present work is to provide reference results for the validation of numerical codes. A simplified configuration has been considered here, which consists of a porous truncated vertical circular cylinder. The (BVP) has been formulated for the inner and the outer domains within the classical potential flow assumptions. A linear porosity model has been considered for the body boundary condition [2][5][7]. The diffraction-radiation problem is then solved with an appropriate eigenfunction expansion.

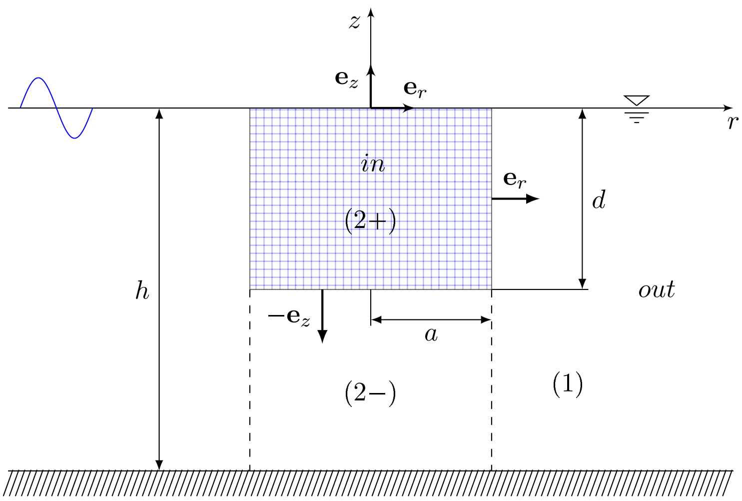

Figure 1: porous circular cylinder, configuration sketch

Mathematical formulation

A cylindrical coordinate system $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ is adopted here with the z-axis pointing upward and $z = 0$ the undisturbed free surface. g is the gravity, h the water depth, ρ the fluid density, ω the wave frequency and $\nu = \omega^2/g$ the infinite depth wave number. The cylinder is supposed to be rigid with only three independent radiation problems to be considered: surge $j = 1$, heave $j = 3$ and pitch $j = 5$ calculated with respect to the centre of waterplane. a stands for the cylinder radius and d its draft. The first order total displacement $\mathbf{H}(\mathbf{R}, \omega)$ can be written as:

$$\mathbf{H}(\mathbf{R}, \omega) = \sum_{j=1,3,5} \xi_j(\omega)\mathbf{h_j}(\mathbf{R}) \tag{1}$$

Where $\mathbf{R} = (r, \theta, z)$ the point position at rest, $\mathbf{h_j}$ the $\mathrm{j^{th}}$ modal displacement vector and $\xi_j(\omega)$ its modal amplitude. As the total displacement, the total fluid potential $\Phi(\mathbf{R}, t) = \Re\left(\phi(\mathbf{R}, \omega)e^{-i\omega t}\right)$ can be expressed

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in frequency domain as:

$$\phi = \phi_I + \phi_D - i\omega \sum_{j=1,3,5} \xi_j \phi_j \tag{2}$$

ϕ_I is the incident potential, ϕ_D the diffraction potential and ϕ_j the radiation potential associated to the j$^{\text{th}}$ mode. The diffraction-radiation potential $\phi_{D/j}$ should satisfy the Laplace equation in the fluid domain, the free-surface boundary condition, the seabed boundary condition and the Sommerfeld condition at the far field:

$$\begin{cases} \nabla^2 \phi_{D/j}^{out/in} = 0 & z < 0 \\ -\nu\phi_{D/j}^{out/in} + \dfrac{\partial \phi_{D/j}^{out/in}}{\partial z} = 0 & z = 0 \\ \dfrac{\partial \phi_{D/j}^{out}}{\partial z} = 0 & z = -h \\ \sqrt{\nu r}(\dfrac{\partial \phi_{D/j}^{out}}{\partial r} - i\nu\phi_{D/j}^{out}) = 0 & r \to +\infty \end{cases} \tag{3}$$

Where the superscript ^{out} (respectively ^{in}) is used for the outer (respectively inner) cylinder region. Concerning the body boundary condition, we consider the cylinder body to be infinitely thin with very fine and numerous pores. Therefore, Darcy's law can be applied. The latter implies that the normal relative velocity is continuous and linearly proportional to the pressure drop through the porous body surface [1]:

$$\frac{\partial \phi_{D/j}^{out}}{\partial n} = \frac{\partial \phi_{D/j}^{in}}{\partial n} = \mathbf{v}.\mathbf{n} + i\sigma\left(\phi_{D/j}^{in} - \phi_{D/j}^{out}\right) \tag{4}$$

$\mathbf{n}$ is the body normal oriented towards the external fluid region, $\mathbf{v}$ the body velocity and σ the porosity parameter. The body is impermeable for $\sigma = 0$ and completely transparent when $\sigma \to \infty$. In our case, σ_b refers to the cylinder bottom porosity coefficient and σ_l to the cylinder sidewall porosity coefficient. Finally, $\mathbf{v}.\mathbf{n} = -\dfrac{\partial \phi_I}{\partial n}$ for diffraction and $\mathbf{v}.\mathbf{n} = \mathbf{h_j}.\mathbf{n}$ for radiation.

Eigenfunction Expansion

The fluid domain is divided into 2 region: region 1 ($r \geqslant a$) denoted by superscript $^{(1)}$ and region 2 ($r \leqslant a$) denoted by superscript $^{(2)}$. For region 2, the superscript $^{(2+)}$ (respectively $^{(2-)}$) indicates that the potential is valid in the upper part $z \geqslant -d$ (respectively lower part $z \leqslant -d$). The first order incident potential is given by:

$$\phi_I = \frac{-ig}{\omega} \sum_{m=0}^{+\infty} \epsilon_m i^m J_m(k_0 r) f_0(z) \cos(m\theta) \tag{5}$$

With $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m > 0$. J_m is the Bessel function of the first kind. The diffraction-radiation potential satisfying equation (3) in region 1 can be written as:

$$\phi_{D/j}^{(1)} = \sum_{m=0}^{+\infty} \left[a_{0m} \frac{H_m(k_0 r)}{H_m(k_0 a)} \frac{f_0(z)}{\sqrt{F_0}} + \sum_{n=1}^{+\infty} a_{nm} \frac{K_m(k_n r)}{K_m(k_n a)} \frac{f_n(z)}{\sqrt{F_n}} \right] \cos(m\theta) \tag{6}$$

Here H_m is the Hankel function of the first kind and K_m the modified Bessel function the second kind. The vertical basis function $f_n(z)$, which is orthogonal, is defined by:

$$f_0(z) = \frac{\cosh(k_0(z+h))}{\cosh(k_0 h)} \quad , \quad f_n(z) = \frac{\cos(k_n(z+h))}{\cos(k_n h)} \tag{7}$$

Where the wave numbers satisfy $\nu = k_0 \tanh(k_0 h) = -k_n \tan(k_n h)$ and $F_n = \int_{-h}^{0} f_n(z)^2 dz$. For $r \leqslant a$, the potential $\phi_{D/j}^{(2)}$ is decomposed into two problems: $\phi_{D/j}^{(2)} = \phi_{D/j}^{(2P)} + \phi_{D/j}^{(2H)}$, with $\phi_{D/j}^{(2P)}$ the particular solution satisfying (3) and the body boundary condition (4) at the cylinder bottom. Consequently, $\phi_{D/j}^{(2H)}$ should verify (3) and the homogeneous body boundary condition (4) at the cylinder bottom:

$$\frac{\partial \phi_{D/j}^{(2H+)}}{\partial z} = \frac{\partial \phi_{D/j}^{(2H-)}}{\partial z} = i\sigma_b \left(\phi_{D/j}^{(2H-)} - \phi_{D/j}^{(2H+)}\right) \tag{8}$$

Following [2], $\phi_{D/j}^{(2H)}$ is expressed as:

$$\phi_{D/j}^{(2H)} = \sum_{m=0}^{+\infty} \left[\sum_{n=1}^{+\infty} b_{nm} \frac{J_m(\mu_n r)}{J_m(\mu_n a)} \frac{g_n(z)}{\sqrt{G_n}} \right] \cos(m\theta) \tag{9}$$

The vertical basis function $g_n(z)$, also orthogonal, is given by:

$$g_n(z) = \begin{cases} \sinh(\mu_n(h-d))\,(\mu_n \cosh(\mu_n z) + \nu \sinh(\mu_n z)) & -d \leqslant z \leqslant 0 \\ (\nu \cosh(\mu_n d) - \mu_n \sinh(\mu_n d)) \cosh(\mu_n(z+h)) & -h \leqslant z \leqslant -d \end{cases} \tag{10}$$

And $G_n = \int_{-h}^{0} g_n(z)^2 dz$ similar to F_n for region 1. The velocity continuity at the cylinder bottom is satisfied thanks to equation (10) whereas the pressure drop condition (8) yields to the following dispersion relation [2] [7]:

$$\mu_n \sinh(\mu_n(h-d))\,(\nu \cosh(\mu_n d) - \mu_n \sinh(\mu_n d)) = i\sigma_b\,(\nu \cosh(\mu_n h) - \mu_n \sinh(\mu_n h)) \tag{11}$$

This dispersion equation has an infinite number of complex roots which are not real or pure imaginary when $\sigma_b \neq 0$. In this case, an iterative scheme was used to evaluate μ_n numerically as suggested by Bao [7]. The particular solution $\phi_{D/j}^{(2P)}$ depend on the problem to be solved. It can be easily found: $\phi_D^{(2P)} = -\phi_I$ for diffraction, $\phi_1^{(2P)} = 0$ for surge and $\phi_3^{(2P)} = \frac{i}{\sigma_b}\mathcal{H}(-z-d)$ for heave, $\mathcal{H}$ being the Heaviside function. Concerning pitch, the pressure drop condition takes the following form:

$$\frac{\partial \phi_5^{(2P)}}{\partial z} = -r\cos(\theta) + i\sigma_b \left(\phi_5^{(2P-)} - \phi_5^{(2P+)} \right) \tag{12}$$

First, the radial coordinate r is expanded using Fourier-Bessel series:

$$r = \sum_{j=1}^{+\infty} \alpha_j \frac{J_1(\kappa_j r)}{J_1(\kappa_j a)} \quad , \quad \alpha_j = \frac{2a}{(\kappa_j a)^2 - 1} \tag{13}$$

κ_j is the j$^{\text{th}}$ root of $\frac{\partial J_1(ar)}{\partial r}$. Using separation of variables, $\phi_5^{(2P)}$ is determined in a similar fashion:

$$\phi_5^{(2P)} = \sum_{j=1}^{+\infty} c_j \frac{J_1(\kappa_j r)}{J_1(\kappa_j a)} g_j^P(z) \cos(\theta) \tag{14}$$

With:

$$c_j = -\frac{\alpha_j}{\kappa_j - i\sigma_b \Delta_j} \quad , \quad \Delta_j = \coth(\kappa_j(h-d)) - \frac{\kappa_j - \nu \tanh(\kappa_j d)}{\nu - \kappa_j \tanh(\kappa_j d)} \tag{15}$$

The vertical basis function $g_j^P(z)$ has the same expression, up to a constant multiplier, as $g_n(z)$ used for the homogeneous solution:

$$g_j^{P+}(z) = \frac{\kappa_j \cosh(\kappa_j z) + \nu \sinh(\kappa_j z)}{\nu \cosh(\kappa_j d) - \kappa_j \sinh(\kappa_j d)} \quad , \quad g_j^{P-}(z) = \frac{\cosh(\kappa_j(z+h))}{\sinh(\kappa_j(h-d))} \tag{16}$$

Matching conditions

The remaining boundary conditions are the velocity continuity at $(r = a, z \leqslant 0)$, the pressure drop at the cylinder wall $(r = a, -d \leqslant z \leqslant 0)$ and the potential continuity across the lower domain $(r = a, -h \leqslant z \leqslant -d)$. Those conditions are written as:

$$\frac{\partial \phi_{D/j}^{(1)}}{\partial r} = \frac{\partial \phi_{D/j}^{(2)}}{\partial r} \qquad r = a \qquad -h \leqslant z \leqslant -d \tag{17}$$

$$\begin{cases} \phi_{D/j}^{(1)} = \phi_{D/j}^{(2)} & r = a \quad -h \leqslant z \leqslant -d \\ \frac{\partial \phi_{D/j}^{(2)}}{\partial r} = \mathbf{v}.\mathbf{e}_r + i\sigma_l \left(\phi_{D/j}^{(2)} - \phi_{D/j}^{(1)} \right) & r = a \quad -d \leqslant z \leqslant 0 \end{cases} \tag{18}$$

Equations (17) and (18) are projected over $f_n(z)$ and $g_n(z)$ to obtain the linear system of the unknown coefficients a_{nm} and b_{nm} (similar to [4]):

$$\int_{-h}^{0} \frac{\partial \phi_{D/j}^{(1)}}{\partial r} \frac{f_n(z)}{\sqrt{F_n}} dz - \int_{-h}^{0} \frac{\partial \phi_{D/j}^{(2H)}}{\partial r} \frac{f_n(z)}{\sqrt{F_n}} dz = \int_{-h}^{0} \frac{\partial \phi_{D/j}^{(2P)}}{\partial r} \frac{f_n(z)}{\sqrt{F_n}} dz \tag{19}$$

$$\int_{-h}^{0}\phi_{D/j}^{(2H)}\frac{g_n(z)}{\sqrt{G_n}}dz+\frac{i}{\sigma_l}\int_{-d}^{0}\frac{\partial\phi_{D/j}^{(2H)}}{\partial r}\frac{g_n(z)}{\sqrt{G_n}}dz-\int_{-h}^{0}\phi_{D/j}^{(1)}\frac{g_n(z)}{\sqrt{G_n}}dz=$$
$$-\int_{-h}^{0}\phi_{D/j}^{(2P)}\frac{g_n(z)}{\sqrt{G_n}}dz-\frac{i}{\sigma_l}\int_{-d}^{0}\frac{\partial\phi_{D/j}^{(2P)}}{\partial r}\frac{g_n(z)}{\sqrt{G_n}}dz+\frac{i}{\sigma_l}\int_{-d}^{0}(\mathbf{v}.\mathbf{e_r})\frac{g_n(z)}{\sqrt{G_n}}dz \quad (20)$$

The linear system is solved for each Fourier mode m for diffraction, only $m = 1$ for surge-pitch and $m = 0$ for heave. Once the potential found, the hydrodynamic forces are calculated by integrating the pressure difference between the two cylinder sides:

$$F_j^{DI}=i\omega\rho\iint_{(S)}(\phi_D^{in}-\phi_D^{out})\mathbf{h_j}.\mathbf{n}dS \quad , \quad \omega^2A_{ij}+i\omega B_{ij}=\rho\omega^2\iint_{(S)}(\phi_i^{in}-\phi_i^{out})\mathbf{h_j}.\mathbf{n}dS \quad (21)$$

Preliminary results and discussion

The obtained solution is compared to available published results for validation. Figure (2) compares horizontal exciting forces on a truncated cylinder with an impermeble bottom [5]. Good agreement is observed with the reference results. On the other hand, figure (3) shows vertical exciting forces for the case where both cylinder bottom and sidewall have the same porosity coefficient $\sigma_b = \sigma_l = \sigma$. b is the nondimensional porosity parameter $b = 2\pi\sigma/k_0$ following Chwang definition [2]. As expected, the hydrodynamics forces are decreasing with the porosity. More detailed results will be presented at the workshop.

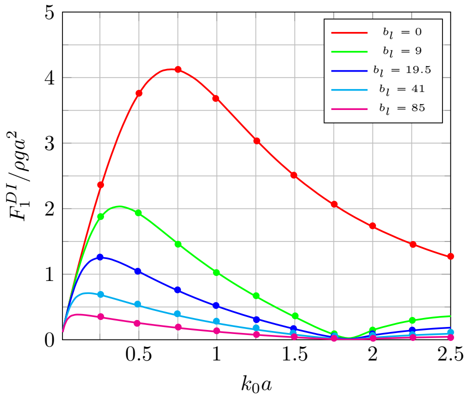

Figure 2: Horizontal force on a porous cylinder, $b_l = 2\pi\sigma_l/k$, $b_b = 0$, $a/d = 0.5$ and $a/h = 0.03$, semi-analytical solution in solid line vs. Zhao [5] in markers

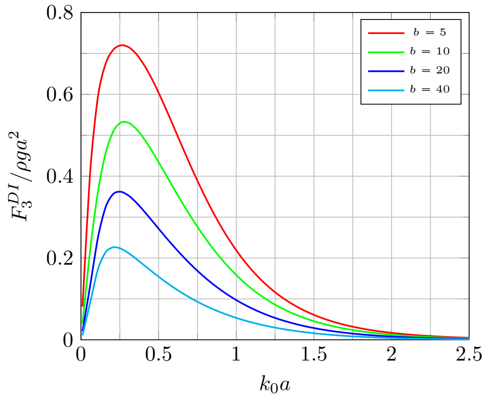

Figure 3: Vertical force on a porous cylinder for different porosity values, $b_l = 2\pi\sigma_l/k_0$, $b_b = 2\pi\sigma_b/k_0$, $b = b_b = b_l$, $a/d = 0.5$ and $a/h = 0.1$

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