Analysis of the generation phase of the upstream waves caused by a ship moving across a depth change

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1 Introduction

Very long upstream waves may be generated when a ship passes over a depth change. The mechanism is found to happen in restricted waters. The waves may however propagate regardless of the water depth, once they are generated. This generation mechanism has been observed for both large and small ships moving at subcritical and supercritical speed, where criticality refers to the shallow water speed at the location. The subcritical generation of the waves has been taking place ever since the modern and very large cruiseferries were introduced the Oslofjord in Norway from 2004 and onwards. A strong forward speed effect has been found where the wave amplitude grows according to the ship speed in a power in the range $\sim 3-4$. A wave height along the shore line of up to 1.4 m has been measured. The 0.5-1 km long waves contribute to a new erosion that is modifying the approximately 40 m deep (on the average) fjord. Marinas as well as piers and small houses on the shore are experiencing a new damage. A first description of the phenomenon is presented in Grue (2017) including observations, mathematical analysis and numerical results. The effect of the ship was represented by a pressure distribution.

In the present contribution we study the generation mechanism in a novel detail including its interpretation. We also model a proper ship geometry.

2 Theory

Let (x_1, x_2) denote horizontal coordinates and y the vertical coordinate with $y = 0$ in the mean free surface. Let t denote time, U forward speed of the ship (along the x_1-direction) and g the acceleration of gravity. The bottom is given by $y = -h + \beta(x_1, x_2)$ where $\beta = \Delta h(1 + \tanh x_1)/2$ gives the depth change.

The potential along the bottom and free surface is connected to the normal velocity along the free surface through the solution of the Laplace equation. The leading, dominant contribution to this solution is obtained by (see Grue, 2017, Section 3)

$$\mathcal{F}(V) = \frac{\omega^2}{g}\mathcal{F}(\phi_F) + \mathcal{F}(V_1), \qquad \mathcal{F}(V_1) = \frac{\mathrm{i}\mathbf{k}}{\cosh kh} \cdot \mathcal{F}(\beta \nabla_1 \phi_B), \tag{1}$$

where $V = (\partial\phi/\partial y)_{y=0}$, $\omega^2 = gk \tanh kh$, ϕ_F denotes the velocity potential at the surface, ϕ_B the velocity potential along the bottom, ∇_1 horizontal gradient, $\beta(\mathbf{x})$ the depth profile, V_1 the effect of the depth change, $\mathcal{F}(\,)$ Fourier transform, $\mathbf{k}$ the wave vector in Fourier space and $k = |\mathbf{k}|$.

The waves are long and amplitude small which means that linear theory is applicable. The linear free surface (kinematic and dynamic) boundary conditions read in their Fourier transformed versions:

$$\frac{\partial \mathcal{F}(\eta)}{\partial t} - \omega^2 \mathcal{F}(\phi_F) = \mathcal{F}(V_1), \qquad \frac{\partial \mathcal{F}(\phi_F)}{\partial t} + g\mathcal{F}(\eta) = 0. \tag{2}$$

Eqs. (2a,b) may be integrated expressing the free surface elevation by the Fourier transformed vertical velocity, $\mathcal{F}(V_1)$, by

$$\mathcal{F}(\eta) = \int_{t_0}^{t} \cos \omega(t-s) \mathcal{F}(V_1)(s) ds, \tag{3}$$

where t_0 denotes the time before the interaction with the depth change. The wave field is obtained by inverse transform of (3). The wave field may be obtained once the vertical velocity given by (1b) is obtained. This requires obtaining of the bottom potential ϕ_B.

3 Analysis of a small depth change

In what follows we assume the four simplifications: 1. the depth change is small ($\Delta h/h << 1$); 2. the potential along the bottom caused by the moving ship, after the depth change, is approximated by the potential before the depth change (where the correction from the depth change is assumed small compared to the 'incoming' ship generated potential); 3. the Froude number based on the ship length is assumed to be small, justifying the rigid lid condition at the surface (before the encounter with the depth change); 4. the back coupling effect by the body is disregarded.

Dipole representation. For illustrative purposes the ship geometry is first represented by a dipole, where a sum of images means that the rigid lid condition is satisfied at the mean free surface at $y = 0$ and along the bottom located at $y = -h$. The potential of a dipole located in $x_1 = 0, x_2 = 0$, moving with speed U, is obtained by

$$\phi(x_1, x_2, y) = UA\frac{\partial}{\partial x_1}\sum_{n=-\infty}^{\infty}\frac{1}{r_n}, \tag{4}$$

where U is the forward speed and $r_n^2 = x_1^2 + x_2^2 + (y + 2nh)^2$. The dipole moment is given by $A = (2\mathcal{V} + 2a_{11})/(4\pi)$ where $\mathcal{V}$ and a_{11} denote the displaced volume and added mass of the body, respectively. The factor of 2 appears because of the double body approximation. The added mass a_{11} of the slender ship is small compared to the volume. The value of the potential at $y = -h$ is obtained by

$$\phi_B = 2UA\frac{\partial}{\partial x_1}\sum_{n=1}^{\infty}\frac{1}{R_n}, \tag{5}$$

where $R_n^2 = x_1^2 + x_2^2 + (2n-1)^2h^2$. Figure 1a illustrates the vertical velocity V_1 in (1b) caused by the dipole located in $(x_1, x_2) = (0, 0)$ moving along the new depth of $h - \Delta h$. The vertical velocity is divided by $A^* = (\mathcal{V} + a_{11})/h^3$. The dipole causes a positive vertical velocity at its forward (positive) pole and a negative vertical velocity at its backward (negative) pole. The vertical velocity is antisymmetrical in the motion direction and symmetrical in the lateral direction. Figures 1c and 1d illustrate the upstream elevation caused by the dipole moving across a depth change at $x_1 = 0$. The speed is $U/\sqrt{gh} = 0.5$ and the time $t = 10\sqrt{g/h}$, clocked after the depth change, where in the figure the dipole is temporarily at $x_1/h = 5$. The substantial depression below the dipole does not contribute to the upstream wave elevation. The periodic domain is $(L_1, L_2) = (160h, 20h)$.

Ship geometry. Assuming that the ship geometry is given by $y = \delta(x_1, x_2)$ it may be shown that the velocity potential given by the moving ship reads

$$k\mathcal{F}(\phi(y)) = \frac{\cosh k(y+h)}{\sinh kh}\Big(\mathcal{F}(W_F) + \mathrm{i}\mathbf{k}\cdot\mathcal{F}(\delta\nabla_1\phi_F)\Big), \tag{6}$$

$$k\tanh kh\mathcal{F}(\phi_F) = \mathcal{F}(W_F) + \mathrm{i}\mathbf{k}\cdot\mathcal{F}(\delta\nabla_1\phi_F) + k\tanh kh\mathcal{F}(\delta W_F), \tag{7}$$

where $W_F = (\partial\phi/\partial n)\sqrt{1 + |\nabla\delta|^2} = -U\mathrm{i}\cdot\nabla\delta = -U\delta_{x_1}$. The normal vector is pointing out of the fluid. The former equation (6) determines $\mathcal{F}(\phi)$ below the geometry (valid for $y < \delta$). The latter equation (7) determines $\mathcal{F}(\phi_F)$ along the free surface and the ship geometry. In the computations the ship geometry is given by $\delta(x_1, x_2) = -d_0(1 - (2x_1/l)^{12} - (2x_2/w)^6)$ where

(l, w, d_0) denotes (length, width, draught). In the computations, $l/h = 15$, $w/h = 2.5$, $d_0/h = 0.5$, and $(L_1, L_2) = (160h, 20h)$.

The evaluation of the vertical velocity V_1 due to the elongated ship moving along the new depth $h - \Delta h$ exhibits a localized upward contribution at the bow and a similar negative contribution at the aft (figure 1b) where the mid position of the ship is at $x_1 = x_{1,0}$ and $x_2 = 0$. The contributions are distributed in the form of two cones, of width similar to the width of the ship. They both emerge when the bow or aft are passing by the depth change. The Fourier transform of the contribution from the bow may be approximated by

$$\mathcal{F}(V_{1,bow}) = \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_1 V_{1,bow}(x_1, x_2) e^{-\mathrm{i}k_1 x_1 - \mathrm{i}k_2 x_2} \simeq V_0 e^{-\mathrm{i}k_1 x_{1,b}} H(x_{1,b}), \tag{8}$$

where V_0 denotes the integrated velocity of the velocity cone, $x_{1,b}$ the x_1-coordinate of the center of gravity of the velocity cone and H the Heaviside function. The contribution from the aft is obtained similarly by

$$\mathcal{F}(V_{1,aft}) = \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_1 V_{1,aft}(x_1, x_2) e^{-\mathrm{i}k_1 x_1 - \mathrm{i}k_2 x_2} \simeq -V_0 e^{-\mathrm{i}k_1 x_{1,a}} H(x_{1,a}), \tag{9}$$

where $x_{1,a} \simeq -x_{1,b}$ denotes the x_1-coordinate of the center of gravity of the velocity cone at the aft. The distance $x_{1,b} - x_{1,a}$ is slightly shorter than the ship length. The effect of the velocity cone at the stern appears at a time $(x_{1,b} - x_{1,a})/U$ earlier than at the aft.

The convolution (3) may be evaluated for a relatively narrow model channel of width L_2 where the wave the motion is averaged across the channel. The time dependencies of $x_{1,b}$ and $x_{1,a}$ are accounted for. The following asymptotic upstream elevation may be obtained: $\eta/[\tilde{\alpha}\mathcal{V}(\Delta h/h^2)/L_2] \sim (Ai(Z)/[(1 - Fr)(4c_0 t/h)^{1/3}]$ where Ai denotes the Airy function, $Z = (x_1 - c_0 t)/(c_0 h^2 t)^{1/3}$ and $c_0 = \sqrt{gh}$. The coefficient $\tilde{\alpha}$ is very close to unity. The asymptotic elevation is plotted in figure 1e for $t\sqrt{g/h} = 40$ and $Fr = 0.5$. The asymptotic wave amplitude grows according to $1/(1 - Fr)$ and becomes singular at $Fr = 1$. Note that the linear theory outlined here is not valid in the transcritical regime.

Interpretation. We note that the vertical velocity V_1 in (1) may be rewritten by

$$\mathcal{F}(V_1) = \frac{\mathrm{i}\mathbf{k}}{\cosh kh} \cdot \mathcal{F}(\beta \nabla_1 \phi_B) \simeq \mathcal{F}(\nabla_1 \beta \cdot \nabla_1 \phi_B + \beta \nabla_1^2 \phi_B) \simeq \mathcal{F}\Big((\nabla_1 \beta - \mathbf{j}) \cdot (grad\, \phi)_{y=\beta}\Big), \tag{10}$$

(assuming $kh << 1$) where the velocity V_1 counteracts the ship-induced normal velocity of the fluid, at $y = -h + \beta$, such that the sum is zero at the new depth. This results in a corresponding vertical velocity at the surface which generates the upstream tsunami. At the bow, the ship induced vertical velocity is negative. If the depth reduces, this results in an upward flux at the position of the bow, and an upstream elevation wave. The vertical velocity takes the opposite direction at the aft. If the depth increases, the downward velocity at the ship bow leads to a negative leading upstream wave.

Finally we note that the Fourier transform of the normal velocity W_F induced by the ship, defined below (7), yields contributions at the bow and the aft, both proportional to the ship's sectional area, separated by a distance corresponding to the ship length. This shows that the ship generated waves are proportional to the volume of the ship.

Reference.

J. Grue (2017) Ship generated mini-tsunamis (2017). J. Fluid Mech. 816, 142-166.

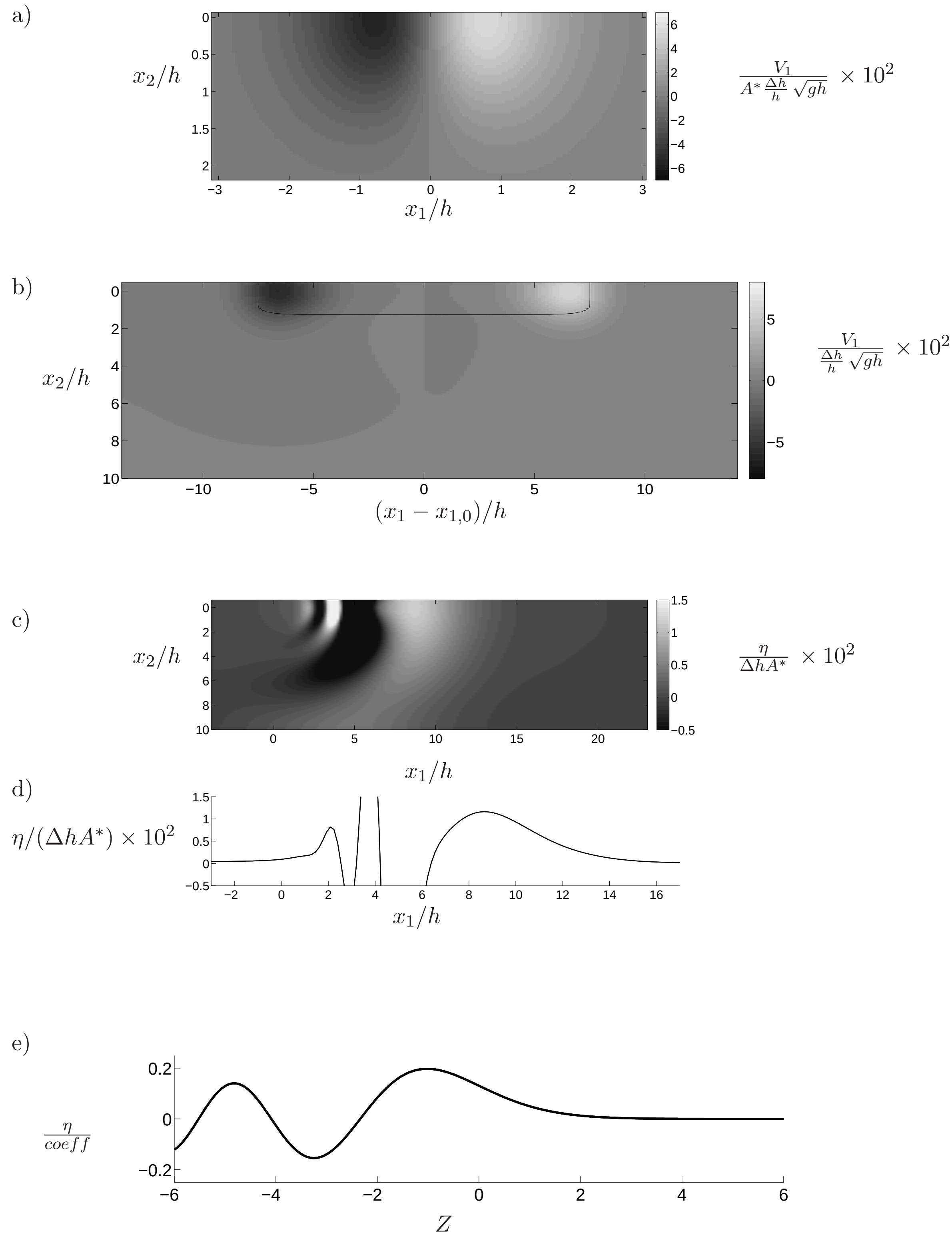

Figure 1: a) $\frac{V_1 \times 10^2}{A^* \Delta h / h \sqrt{gh}}$ for a dipole. b) $\frac{V_1 \times 10^2}{\Delta h / h \sqrt{gh}}$ for a ship (with $\mathcal{V}/h^3 = 13.1$). c) Elevation $\frac{\eta}{A^* \Delta h} \times 10^2$ for the dipole running at $U/\sqrt{gh} = 0.5$ across a depth change at $x_1 = 0$, at time $t = 10\sqrt{h/g}$ after the depth change. d) Same as b) but section along $x_2 = 0$. e) Upstream elevation due to a ship in a channel of width L_2. $t\sqrt{g/h} = 40$, $Fr = 0.5$, and $coeff = \tilde{\alpha}\mathcal{V}(\Delta h/h^2)/L_2$.