# Flexural gravity wave blocking in a two-layer fluid

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## Highlights

- Flexural gravity wave blocking in a two-layer fluid is studied.
- The condition for wave blocking is obtained analytically for surface and internal modes in water of finite depth and the associated results in the shallow water limit are derived as special cases.
- The dependence of the compressive force acting on the ice-covered surface and period of the blocked waves are illustrated graphically.

## 1. Introduction

Flexural gravity waves in an ice-covered two-layer fluid is studied with an emphasis on wave blocking due to the compressive force. This work extends the results found earlier by [1] in an ice-covered homogeneous fluid. The wave blocking phenomena is observed in the internal mode as well as the surface mode. Moreover, there exists situations where the internal mode, at the interface between two layers of fluid, propagates faster than surface mode. This is contrary to the usual result where the surface mode is faster than the internal mode. Furthermore, the dependency of compressive force and time-period of blocked waves are shown pictorially for two different cases, namely finite upper layer fluid and shallow water approximation.

## 2. Mathematical formulation

In the present study we consider the small amplitude response of a floating thin elastic plate, the standard model for sea ice and very large floating structures. The thin elastic plate is assumed to be of infinite extent and floating on the mean free surface of water of finite constant depth. The problem is two-dimensional with the $x$-axis horizontal and the $z$-axis vertically downward (Fig. 1). The fluid domain consists of two different immiscible fluids of density $\rho_1$ and $\rho_2$ such that $\rho_2 > \rho_1$ and we define $s = \rho_1/\rho_2$. Our interest here is to understand in detail the properties of the flexural gravity waves which can propagate in such a system. The upper fluid occupies the region $-\infty < x < \infty$, $0 \le z \le h$ and the lower fluid occupies the region $-\infty < x < \infty$, $h \le z \le H$. The fluids are assumed to be inviscid, incompressible and both fluids motion are irrotational which ensures the existence of the velocity potentials $\Phi_j(x, z, t)$ for $j = 1, 2$ which satisfy the two-dimensional Laplace equation given by

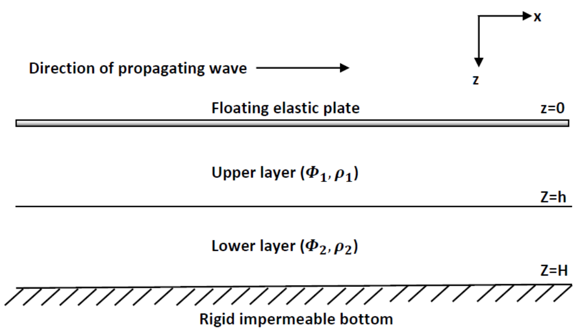


Figure 1: Schematic diagram of the physical problem

$$\nabla^2 \Phi_j = 0, \quad j = 1, 2, \tag{1}$$

in the upper and lower layer fluid respectively. The linearized kinematic boundary condition on the plate covered surface is given by

$$\frac{\partial \eta_1}{\partial t} = \frac{\partial \Phi_1}{\partial z} \quad \text{on} \quad z = 0, \tag{2}$$

where $\eta_1(x, t)$ is the deflection of the floating elastic plate. In the presence of an in-axis compressive force $N$ acting along the $x$-axis of the homogeneous floating elastic plate, the linearized plate covered dynamic boundary condition on the mean free surface is given by [2])

$$\left(EI\frac{\partial^4}{\partial x^4}+N\frac{\partial^2}{\partial x^2}+\rho_p d\frac{\partial^2}{\partial t^2}+\rho_1 g\right)\eta_1=\rho_1\frac{\partial \Phi_1}{\partial t} \quad \text{on } z=0, \tag{3}$$

where $EI = Ed^3/(12(1-\nu^2))$ is the flexural rigidity, E is Young's modulus, $\nu$ is Poisson's ratio, $d$ is the thickness of the floating plate and $\rho_p$ is the plate density. The linearized dynamic and kinematic boundary conditions at the the mean interface are

$$\rho_2\left\{g\eta_2-\frac{\partial \Phi_2}{\partial t}\right\}-\rho_1\left\{g\eta_2-\frac{\partial \Phi_1}{\partial t}\right\}=0 \quad \text{and} \quad \frac{\partial \eta_2}{\partial t}=\frac{\partial \Phi_j}{\partial z} \quad \text{for} \quad j=1,2, \quad \text{on} \quad z=h, \tag{4}$$

where $\eta_2(x,y,t)$ is the interface elevation. Finally, at the rigid bottom the boundary condition is given by

$$\frac{\partial \Phi_2}{\partial z}=0, \quad z=H. \tag{5}$$

## 3. Characteristics of wave motion

Characteristics of flexural gravity wave motion in surface and internal modes are investigated by analyzing the dispersion relation associated with the problem by assuming plane wave solutions of the forms

$$\eta_1=\mathrm{Re}\{a_1\mathrm{e}^{\mathrm{i}(kx-\omega t)}\}, \quad \text{and} \quad \eta_2=\mathrm{Re}\{a_2\mathrm{e}^{\mathrm{i}(kx-\omega t)}\}, \tag{6}$$

where $k$ is the wave number, $\omega$ is the angular frequency, $a_1$ and $a_2$ are the amplitudes of the plate deflection and interface elevation respectively. Thus, the associated velocity potentials $\Phi_1$ and $\Phi_2$ satisfying (1) along with the boundary conditions (2) and (5) are given by (see [3])

$$\Phi_1=(A\cosh kz-\mathrm{i}\omega a_1 k^{-1}\sinh kz)\mathrm{e}^{\mathrm{i}(kx-\omega t)} \quad \text{and} \quad \Phi_2=B\cosh k(H-z)\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{7}$$

with $A$ and $B$ being the unknown constants to be determined. Using the boundary conditions (4) in (7), the constants A, B and $a_2$ are obtained in terms of $a_1$ which in turn yields the velocity potential in upper layer fluid as

$$\Phi_1=-\frac{\mathrm{i}\omega}{k}(\mu\cosh kz+\sinh kz)\eta_1 \quad \text{with} \quad \mu=-\frac{s+\coth kh\{\coth k(H-h)-(1-s)gk/\omega^2\}}{s\coth kh+\coth k(H-h)-(1-s)gk/\omega^2} \tag{8}$$

$$\text{and} \quad \frac{a_2}{a_1}=\frac{s}{\sinh kh\{s\coth kh+\coth k(H-h)-(1-s)gk/\omega^2\}}. \tag{9}$$

Under the assumption that the plate thickness is very small compared to the wavelength, i.e., assuming the elastic restoring force to be much stronger than the inertial force, we neglect the term $\rho_p d\omega^2/\rho_1 << 1$ following [3]. This assumption simplifies our equations and physically is sensible because the term needs to be small if the elastic plate floats and is thin. Thus, using (3) and (8), the dispersion relation for the flexural gravity wave motion in two-layer fluid of finite depth is obtained as

$$A_1\omega^4-B_1\omega^2+C_1=0, \tag{10}$$

where $A_1 = s+\coth kh\coth k(H-h)$, $B_1 = gk[(1-s)\coth kh + A_2\{s\coth kh+\coth k(H-h)\}]$, $C_1 = A_2(1-s)g^2k^2$ and $A_2 = Dk^4-Qk^2+1$ with $D = EI/(\rho_1 g)$ and $Q = N/(\rho_1 g)$. It may be noted that in [3] effect of compression was not included in their two-layer fluid model. The dispersion relation in Eq. (10) is a quadratic equation in $\omega^2$ whose solution yields the wave frequencies in the surface and internal modes and are denoted as $\omega_+$ and $\omega_-$ respectively. Thus,

$$\omega_\pm^2=\frac{B_1\pm\sqrt{B_1^2-4A_1C_1}}{2A_1} \quad \text{and} \quad c_g^\pm=\frac{\omega_\pm^2\frac{dB_1}{dk}-\omega_\pm^4\frac{dA_1}{dk}-\frac{dC_1}{dk}}{2\omega_\pm(2A_1\omega_\pm^2-B_1)}, \tag{11}$$

where $c_g$ is the group velocity with subscripts $\pm$ corresponding to waves in surface and internal modes respectively. As discussed in [1], occurrences of blocking are illustrated in Fig. 2. Blocking points are readily identified by the maximum (primary blocking) and minimum (secondary blocking). The behaviour of our system is complicated and we make here a further simplification. Under the assumption of deep water approximation,

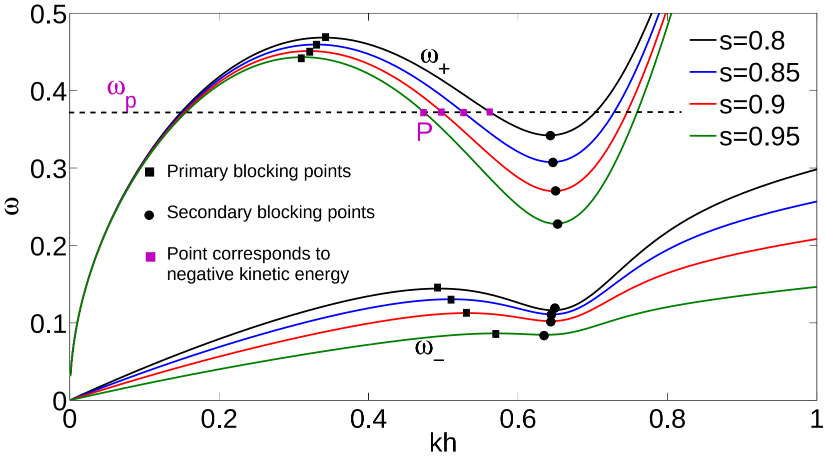


Figure 2: Dispersion graphs for both the waves at surface and interface are depicted for different values of density ratio $s$ with $Q/\sqrt{D} = 1.95$ and $h = 10$. Wave blocking, characterized by the occurrences of maxima and minima, is observed in both the modes. Rigid black squares corresponds to primary blocking, whereas rigid black circles to secondary blocking. When density ratio increases, i.e., difference of densities is small, slope of the dispersion curve for surface wave (movement of point $P$ along dispersion curves for different values of density and fixed incoming wave frequency $\omega_p$) in between two blocking points increases where as reverse pattern is observed for waves in internal mode. This results in higher rate of negative kinetic energy propagation for surface waves which is due to the transfer of such energy from internal waves.

assuming $|k|h >> 1$ and $|k|(H-h) >> 1$, (11) yields

$$\omega_+^2 = (Dk^4 - Qk^2 + 1)g|k|, \quad \omega_-^2 = \frac{(1-s)g|k|}{1+s}. \tag{12}$$

The dispersion graph is plotted in Fig. 3 and the regions where internal mode travels faster than the surface mode are identified. On the other hand, when upper layer depth is comparable with the wavelength ($kh = O(1)$) and lower layer water depth is large ($k(H-h) >> 1$), the dispersion relation (11) yields

$$\omega_+^2 = \frac{(Dk^4 - Qk^2)(s \coth kh + 1) + 1 + \coth kh}{\coth kh + s} gk, \quad \omega_-^2 = \frac{(1-s)A_2 gk}{(1 + s\coth kh)A_2 + (1-s)\coth kh}. \tag{13}$$

Wave blocking occurs in both the wave modes when no wave energy propagation occurs. Since, the rate at which wave energy propagates is proportional to group velocity of the wave, wave blocking is identified

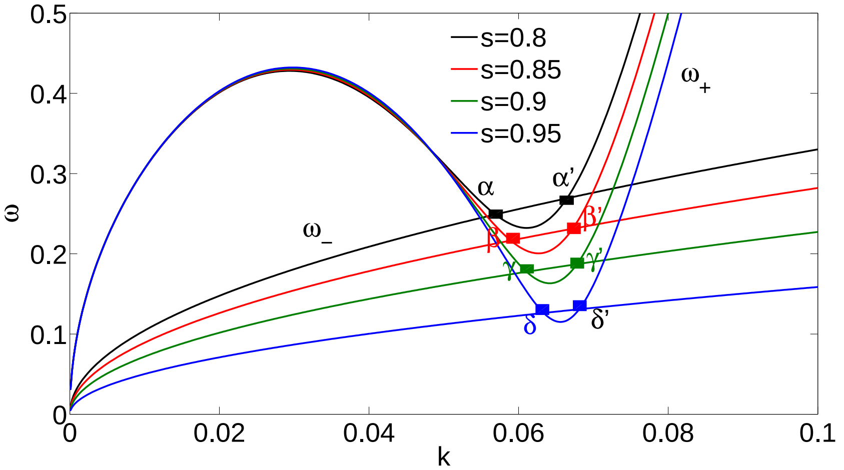


Figure 3: Under deep water approximation, the regions $(\alpha, \alpha')$, $(\beta, \beta')$, $(\gamma, \gamma')$ and $(\delta, \delta')$ correspond to the regions where internal mode travels faster than the surface mode.

mathematically by $d\omega_{\pm}/dk = 0$. For surface waves, in particular, the condition on wave blocking yields the following expression for compressive force:

$$Q = \frac{\left\{5pr - kh(1-s^2)(1-\coth^2 kh)\right\} Dk^4 + qr - kh(1-s)(1-\coth^2 kh)}{k^2 \left\{3pr - kh(1-s^2)(1-\coth^2 kh)\right\}} \tag{14}$$

where $\quad p = s\coth kh + 1, \quad q = 1 + \coth kh \quad$ and $\quad r = s + \coth kh.$

It is to be noted that, for fixed frequency, the above expression provides two different values of $Q$ corresponding to primary and secondary blocking. Substituting the above expression into (13), the following relation between blocking frequency and wave number is obtained for both the surface and internal modes, respectively, as

$$\omega_+^b = \sqrt{\frac{2pq - kh(1-s)(1-\coth^2 kh) - 2p^2 Dk^4}{3pr - kh(1-s^2)(1-\coth^2 kh)} gk} \quad \text{and} \quad \omega_-^b = \sqrt{\frac{-\hat{B} + \sqrt{\hat{B}^2 - 4\hat{A}\hat{C}}}{2\hat{A}}}, \tag{15}$$

where $\hat{A} = 2(Dk^4-1)(1+s\coth kh)^2 - (1-s)\left\{\tau + (1+2s\coth kh)\coth kh\right\}$, $\tau = \coth kh - kh(1-\coth^2 kh)$, $\hat{B} = gk(1-s)\left\{(1-s)(\tau + 2\coth kh) - 4(Dk^4-1)(1+s\coth kh)\right\}$ and $\hat{C} = 2g^2k^2(1-s)^2(Dk^4-1)$, and the corresponding compressive force is calculated from the following relation:

$$(Dk^4 - Qk^2 + 1)^2 + \frac{(1-s)\tau}{(1+s\tau)}(Dk^4 - Qk^2 + 1) + 2\frac{(1-s)}{(1+s\tau)}(Dk^4 - 1)\coth kh = 0. \tag{16}$$

Figures 4 and 5 demonstrate the dependencies of compressive force and time-period of blocked waves for surafce and internal modes respectively.

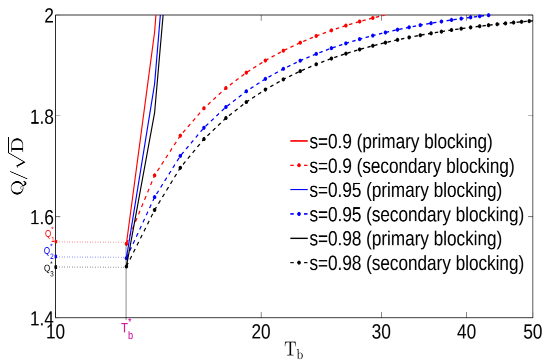


Figure 4: Dependency between compressive force and time-period of blocked surface waves are demonstrated for different values of density ratio. Rigid lines correspond to primary blocking (related to incident wave) and dash-dot lines are for secondary blocking (related to waves with negative kinetic energy). Point of inflexion where wave blocking initiates, is represented by the points $(Q_i^*, T_b^*)$, $i = 1, 2, 3$ for $s = 0.9$, $0.95$ and $0.98$, respectively. The compressive force required to block a wave of specific time-period is represented by the points on the curves. However, waves having time-period less than $T_b^*$ never gets blocked by the action of compressive force. Also, depending upon the density ratio, a minimum of compressive force is required to achieve wave blocking, e.g. $Q_1^*$ for $s = 0.9$.

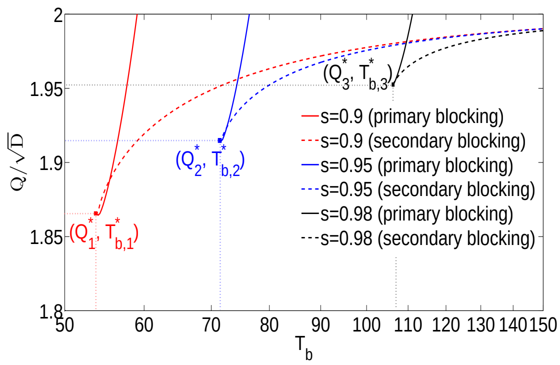


Figure 5: Internal wave time-period and compressive force acting on the floating plate dependency is pictorially depicted for different values of density ratio. The pattern of the graphs are similar in nature as observed in Fig. 4. Higher density ratio requires high compressive force to initiate blocking for waves with higher time-period. The compressive force $Q_3^*$ corresponding to point of inflexion for waves in stratified fluid having density ratio $s = 0.98$ is higher than $Q_2^*$ (for $s = 0.95$) and $Q_1^*$ (for $s = 0.9$). Corresponding time-period $T_{b,3}^*$ is also higher than $T_{b,2}^*$ and $T_{b,1}^*$.

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