A primary analysis of water wavefronts

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In the fore part of a train of progressive waves generated on water surface by harmonic motions of a wavemaker, the transient waveform ahead of wave train is often called wavefront. A primary analysis based on the method of contour integral in complex wavenumber plane, different from that used by Miles (1962), is performed here to study the exact behavior of wavefront. We define wavefront function as the wave amplitude which envelopes the forerunners in the wavefront. It is shown that the starting position of wavefront propagates at the very group velocity and has an amplitude value different from 1/2 that predicted by Miles. Three classes of waves in different region are then demystified and their interesting features are revealed.

1 Introduction

A lot of studies on wave generations by different wavemakers in a wave flume or a basin have been carried out, as regularly reported by ITTC. Typical analyses such as that by Joo *et al.* (1990) are more focused on singularities of flow on wavemakers on one side, and steady-state waves at some distance from wavemaker on the other, or guidelines by ITTC for selecting wavemaker types as function of desired wave characteristics. However, few studies are on transient wavefront generated by an arbitrary excitation or harmonic motions of wavemaker. The classical one by Miles (1962) started with the formal linear solution of Cauchy-Poisson type, to give the linear wave elevation by a time convolution integral of the product of excitation and the memory function which is expressed by a wavenumber integral. The approximation of the memory function obtained by the method of stationary phase is then used in the convolution integral to obtain asymptotic representation of wave elevations including the wave envelope. Miles' results are then valid for waves observed at a position far from wavemaker. The recent results by Eatock Taylor *et al* (1994) show that the wave envelope of Miles does not fully wrap up linear wave elevations in the wavefront nor transient region behind.

New analysis based on the method of contour integral in complex wavenumber plane is performed here. The linear wave elevation generated by harmonic motions of a flexible vertical plate, same as that studied by Eatock Taylor *et al* (1994) but in deepwater is considered. The formal solution represented by a single wavenumber integral (after integration of the time convolution) is reformulated in complex plane and by making use of a new integral variable. The contour integral in complex plane permits us to get contributions from poles and remaining integral along the imaginary axis. The residues represent the steady-state oscillations and an exponential decreasing term behind the wavefront. The integral along the imaginary axis with an exponentially decreasing integrand representing oscillations in wavefront is called wavefront function. This wavefront function is further written in asymptotic expressions using the complex error function. We define three classes of waves in different regions which are the wavefront starting from the position associated with wave group velocity, the steady-state wave train at some distance behind the wavefront, and the transient wave train in between. Furthermore, the wave envelope at the wavefront is shown to be larger than 1/2 that predicted by Miles (1962).

2 Wave elevations due to harmonic motions of wavemaker

We consider a semi-infinite fluid of gravity $g = 1$ limited at top by the free surface and use a Cartesian coordinate system (o, x, z) located at the mean surface with axis oz pointing upwards. The flexible vertical plate is located at $x = 0$ and oscillating with horizontal velocity $\mathcal{A}(z)\sin(\omega t)$

with amplitude $\mathcal{A}(z) = A\omega \exp(k_0 z)$ along the plate, frequency ω and wavenumber k_0. The wave elevation for $x > 0$ is obtained by the classical method based on Fourier transform in Dai (1994) and is written as :

$$\eta^I(x,t) = -A\sin(k_0x)\cos(\omega t) - A\frac{2k_0}{\pi}\fint_0^\infty \frac{\cos(kx)\cos(\beta t)}{k^2-k_0^2}\,dk \tag{1}$$

where $\fint$ stands for the principal value in the sense of Cauchy. The dispersion relation imposes :

$$\omega = \sqrt{k_0} \quad \text{and} \quad \beta = \sqrt{k}$$

The integral (1) can be rewritten in a compact form

$$\eta^I(x,t)/A = \Re\{\eta(x,t)\} \quad \text{with} \quad \eta(x,t) = \int_0^\infty \frac{k_0}{\pi(k_0+k)} \frac{e^{i(kx+\beta t)} + e^{i(kx-\beta t)}}{k_0 - k}\,dk \tag{2}$$

in which $\int$ stands for the integration along the real axis but circumventing above the pole $k = k_0$. Now we consider the change of variables below

$$k = u^2t^2/4x^2\ , \quad u_0 = 2\omega x/t \quad \text{and} \quad \tau = t^2/4x \tag{3}$$

in (2). We have :

$$\eta(x,t) = e^{-i\tau}\int_0^\infty F(u_0,u)\frac{e^{i(1+u)^2\tau} + e^{i(1-u)^2\tau}}{u_0-u}\,du \tag{4}$$

with

$$F(u_0,u) = \frac{2u_0^2 u}{\pi(u_0^2+u^2)(u_0+u)} \tag{5}$$

associated with two oscillatory functions $\exp[i(1+u)^2\tau]$ and $\exp[i(1-u)^2\tau]$. The contour associated with the first along which the integrand is steepest descent is easy to find. The contour associated with the second is more complex due to the multiple values of $(1-u)^2$ for $u \in (0,2)$ and the location of poles. The detail is too lengthy to present in this summary. The final result is

$$\eta(x,t) = -i\left[e^{-\omega t - ik_0x} + e^{i(k_0x-\omega t)}\right] H(1-u_0) + E(u_0,\tau)e^{-it^2/4x} \tag{6}$$

with $H(\cdot)$ the Heaviside function and

$$E(u_0,\tau) = \frac{1}{2}\int_0^\infty f(u_0,\tau,y)e^{-y\tau}\,dy \tag{7}$$

involving

$$\begin{aligned} f(u_0,\tau,y) = -ie^{i\tau}&\left\{\frac{F(\sqrt{1+iy}-1)}{\sqrt{1+iy}(\sqrt{1+iy}-1-u_0)} + \frac{F(1-\sqrt{1+iy})}{\sqrt{1+iy}(\sqrt{1+iy}-1+u_0)}\right\} \\ &+ (1-u_0)e^{-i\pi/4}\frac{F(1+\sqrt{iy})+F(1-\sqrt{iy})}{\sqrt{y}[y+i(1-u_0)^2]} - \frac{F(1+\sqrt{iy})-F(1-\sqrt{iy})}{y+i(1-u_0)^2} \end{aligned} \tag{8}$$

The result (6) is identical to (2) and uniformly valid with respect to $u_0 = 2\omega x/t$ and regardless of $x > 0$ which should be large in Miles' development. The first part has an exponential decreasing term with respect to ωt and the steady-state oscillations $\sin(k_0x - \omega t)$ as we take the real part. The steady-state waves are present only for $u_0 < 1$, i.e. $x < t/2\omega = c_g t$ with $c_g = 1/2\omega$ the group velocity according to the definitions (3) so that the line $x = c_g t$ is the front of wave train. Since only the second part is present for $x > c_g t$, i.e. ahead of wave train, the amplitude function $E(u_0,\tau)$ is called wavefront function. Of course, the second part is also present behind the wavefront and oscillatory described by $\exp(-it^2/4x)$.

3 Wavefront function and its approximation in wavefront

The transient waves with oscillations of form $\exp(-it^2/4x)$ have an amplitude wavefront function defined by the integral (7) with an exponential decreasing integrand. The integrand (8) is a smooth function except for $u_0 = 1$ and at $y = 2$ where $F(1-\sqrt{iy})$ is singular. This singularity is removable and the integration can be performed accurately by using a classical quadrature method. The singularity of order $O(y^{-1/2})$ for $y \to 0$ is integrable. Indeed, the singular behavior of $f(u_0, \tau, y)$ near $y = 0$ needs some special treatments.

In fact, for large values of τ and $u_0 > 0$, the major contribution of (7) comes from the values of $f(u_0, \tau, y)$ near $y = 0$. It is understood that the first term involving $\sqrt{1+iy}$ is of higher order and the Taylor expansion of $F(1 \pm \sqrt{iy})$ at $y = 0$ is used to approximate $f(u_0, \tau, y)$ given by (8). The approximation of $f(u_0, \tau, y)$ is then introduced in (7) to obtain :

$$E(u_0, \tau) \approx -i\pi \{\text{sign}(u_1)F_0 - \pi|u_1|(u_1F_2/2 - F_1)\} e^{iu_1^2\tau}\mathbf{erfc}(\sqrt{iu_1^2\tau}) + e^{i\pi/4}(u_1F_2/2 - F_1)\sqrt{\pi/\tau} \quad (9)$$

in which $\mathbf{erfc}(\cdot)$ stands for the complementary error function defined in Abramowitz & Stegun (1967), $u_1 = 1 - u_0$ and

$$F_0 = F(u_0, 1) \; ; \quad F_1 = F'(u_0, 1) \; ; \quad F_2 = F''(u_0, 1) \quad (10)$$

with the function $F(u0, u)$ given by (5).

The analytical expression (9) is a very good approximation of the wavefront function (7) since the error is of order $O(\tau^{-3/2})$. The function $\exp(z^2)\mathbf{erfc}(z)$ being a smooth function, the wavefront function $E(u_0, \tau)$ is smooth too except there is a step for the imaginary part when u_0 crossing $u_0 = 1$. The asymptotic values of wavefront function for large u_0 and small u_0 are

$$E(u_0, \tau) \approx O(u_0^{-2}\tau^{-1/2}) \quad \text{for} \quad u_0 \gg 1 \quad \text{and} \quad E(u_0, \tau) \approx O(u_0^2) \quad \text{for} \quad u_0 \ll 1 \quad (11)$$

In the vicinity of $u_0 = 1$, we have :

$$E(u_0, \tau) \approx -(i/2)\text{sign}(\epsilon) + (1/4)e^{i\pi/4}/\sqrt{\pi\tau} + O(\epsilon) = (1/\sqrt{2\pi\tau}, \pm 1 + 1/\sqrt{2\pi\tau})/2 + O(\epsilon) \quad (12)$$

for a positive $\epsilon \to 0$. It is worth noting that the approximation (9) is not valid for $u_0 \to 0$ or for

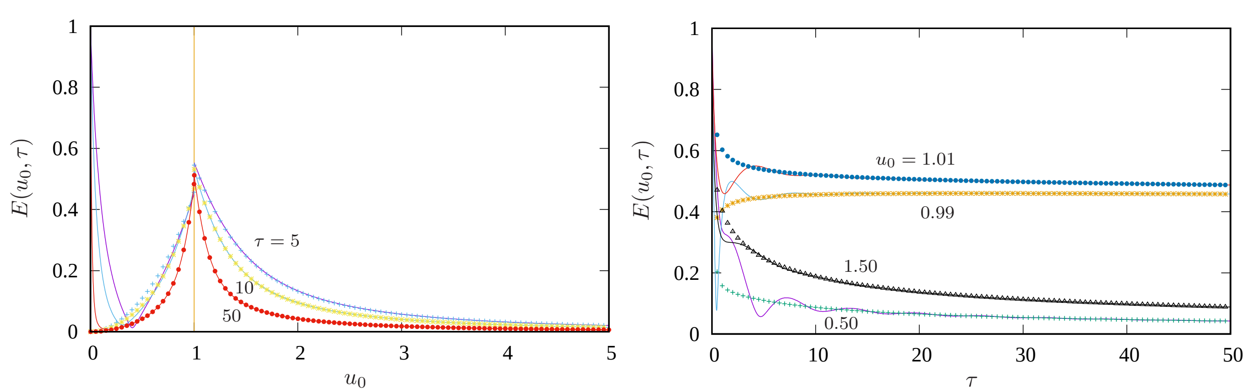

Figure 1: Wavefront function $E(u_0, \tau)$ on the left with varying u_0 and on the right with varying τ

small values of τ. The results of wavefront function by using the exact formulation (7) and those of approximation (9) are depicted on Figure 1. The curves represent exact values while symbols depict the approximations on the left part (for several values of τ) and on the right part (for several values of u_0) of the figure.

4 Three classes of waves

The waves defined by (6) can be regrouped as the sum of three classes

$$\eta(x,t) = \mathcal{S}(k_0 x - \omega t)H(t - 2\omega x) + \mathcal{B}(t^2/4x)H(t - 2\omega x) + \mathcal{F}(t^2/4x)H(2\omega x - t) \tag{13}$$

with the steady-state part of harmonic oscillations

$$\mathcal{S}(k_0 x - \omega t) = -ie^{i(k_0 x - \omega t)} \tag{14}$$

the transient part behind the wavefront

$$\mathcal{B}(t^2/4x) = -ie^{-\omega t - ik_0 x} + E(u_0, \tau)e^{-it^2/4x} \tag{15}$$

and the wavefront

$$\mathcal{F}(t^2/4x) = E(u_0, \tau)e^{-it^2/4x} \tag{16}$$

These three classes of waves are depicted on Figure 2 along the wave propagation path x/λ from

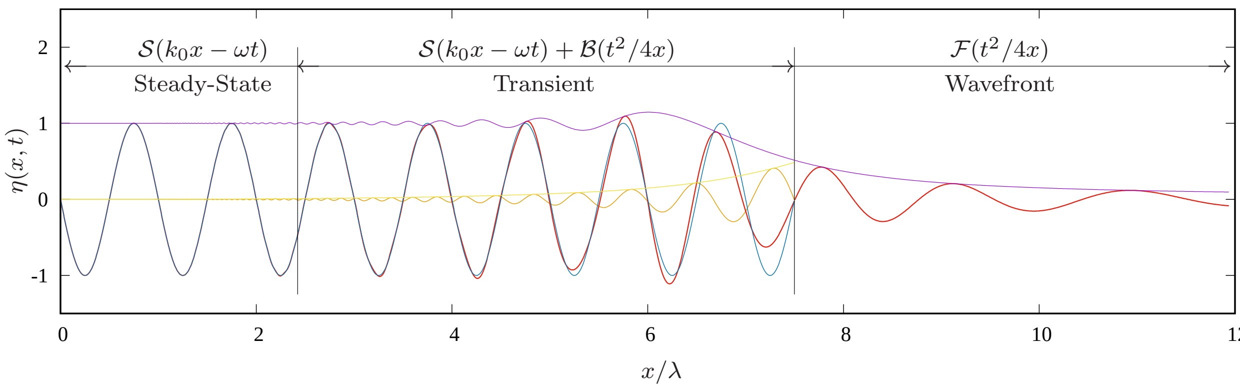

Figure 2: Waves at the instant $t = 15(2\pi/\omega)$ along $x(k_0/2\pi)$ from 0 to 12 for $k_0 = 1.5$

0 to 12 with $\lambda = 2\pi/k_0$ the wavelength at the instant $t = 15T$ with $T = 2\pi/\omega$ the period. Three wave regions are distinguished. The wavefront (red line) represented uniquely by $\mathcal{F}(t^2/4x)$ starting from the position $x = c_g t$ is oscillatory but with waves of wavelength increasing with x. Depending on the observation instant t, there may have one or several forerunners in wavefront. Behind the wavefront, a train of progressive waves (red line) with varying amplitude which is the sum of regular waves $\mathcal{S}(k_0 x - \omega t)$ with constant wave amplitude (blue line), and transient waves $\mathcal{B}(t^2/4x)$ with fast decreasing amplitude for decreasing x (yellow line). Till some distance where the amplitude of $\mathcal{B}(t^2/4x)$ is smaller than 0.01 (if we accept the criterion to neglect waves of amplitude smaller than 1% that of steady-state waves), wave train becomes regular (steady-state, blue line and red line are superposed). Furthermore, the envelope of waves (violet line) over three regions and that (light yellow line) of transient waves $\mathcal{B}(t^2/4x)$ are drawn.

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