Nonlocal Model for Deep Water Waves of a Potential Flow

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I. INTRODUCTION

Deep-water waves differ qualitatively from other wave phenomena. No wave equation exists, which means that the wave is essentially a surface phenomenon, governed by the forces on the surface particles. The bulk fluid is a medium for transmitting a continuous pressure field that adjusts itself to the gravitational motions of the surface particles. This pressure adjustment is instantaneous in the incompressible fluid.

Stokes [1] initiated the theory of nonlinear deep-water waves. Three branches of development for strongly nonlinear theory have grown from seminal papers. (i) The highest Stokes wave [2]. (ii) The instability of Stokes waves [3]. (iii) Breaking of deep-water waves [4].

The early nonlinear theories in 2D have now grown into full 3D theories. However, a long-standing dilemma is that of dimensional reduction: No consistent scheme exists for reducing the full nonlinear 3D problem to a 2D analysis, or a reduction of the full nonlinear 2D problem to a 1D analysis. Different attempts in this direction exists, and prominent among these are: (i) Nonlinear Schrödinger-type equations for deep-water wave groups. (ii) Hamiltonian formulation of deep-water waves.

The present paper offers a fully consistent scheme of reducing the spatial dimension of nonlinear deep-water wave theory by introducing the concept of fractional derivatives along the free surface. Starting from the 2D Euler equations for an incompressible potential flow, a one-dimension model describing deep water surface waves is derived. Similar to the Shallow Water case, the z-dependence of the dependent variables is integrated out and a set of two equations for the surface velocity and the surface elevation remains. The model is nonlocal and can be formulated in conservative form, describing waves over an infinitely deep layer. Finally, numerical solutions are presented for different initial conditions. Coherent wave trains are unstable due to the Benjamin-Feir instability [3] and localized solutions are obtained.

II. THEORY

A. Potential flow

For brevity, the derivation of the model is presented only in two spatial dimensions, x, z. It can be straightforward extended to the 3D case.

We are looking for solutions of the non-dimensional Euler eqs. for an incompressible fluid in the half space $-\infty < z \le h(x, y, t)$. In lateral direction we assume periodic boundary conditions (b.c.) on $0 \le x \le \Gamma$. We use the time scale $\tau = \sqrt{\ell/g}$ with an arbitrary length scale ℓ. We assume that the flow field can be derived from a potential

$$\vec{v} = \nabla\Phi \tag{1}$$

where Φ is a solution of the Laplace eq., implying

$$\nabla^2\vec{v} = 0 \ . \tag{2}$$

Let $u_h(x,t) = v_x(x, z = h(x,t), t)$ be the horizontal velocity component at the free surface located at $z = h$. From the Euler eqs. one derives

$$\partial_t u_h = -\partial_x\left[\frac{1}{2}u_h^2 + h\right] \tag{3}$$

where $P(z = h) = h$. Eq. (3) serves as a boundary condition for (2) at $z = h(x,t)$. The surface function h is determined by the kinematic b.c.

$$\partial_t h = v_z|_{z=h} - v_x|_{z=h}\partial_x h \ . \tag{4}$$

For the following treatment, it is of advantage to formulate (4) also in conservative form

$$\partial_t h = -\partial_x S \tag{5}$$

with the flux

$$S(x,t) = \int_{-\infty}^{h(x,t)} dz \; v_x(x,z,t) \ . \tag{6}$$

For an infinitely deep layer, the asymptotic b.c. $\vec{v} = 0$ for $z \to -\infty$ must hold. Hence the general solution of (2) reads

$$v_x = \sum_k u_k(t) \mathrm{e}^{|k|z} \mathrm{e}^{ikx} + c.c. \tag{7}$$

with c.c. as the complex conjugate.

B. Nonlocal expansion

Considering (3) and (5) as evolution eqs., the original 2D problem is reduced to a 1D system in the horizontal coordinate x only. However, to evaluate the flux (6) one needs to know $v_x(x,z,t)$. Taking (7), eq. (6) reads

$$S(x,t) = \sum_k \frac{\mathrm{e}^{|k|h(x,t)}}{|k|} u_k(t) \mathrm{e}^{ikx} + c.c. \tag{8}$$

To determine the amplitudes u_k from u_h, we evaluate (7) at $z = h$ and expand the exponential function up to a given order h^N:

$$u_h = \sum_k u_k(t) \sum_{n=0}^{N} \frac{(|k|h)^n}{n!} \mathrm{e}^{ikx} + c.c. \tag{9}$$

Introducing the fractional differential operator of order h^N:

$$\hat{L}_N(h) = \sum_{n=0}^{N} \frac{h^n (-\partial_{xx})^{n/2}}{n!} \tag{10}$$

with the Fourier representation

$$(-\partial_{xx})^{n/2} \longrightarrow |k|^n \tag{11}$$

eq. (9) can be written as

$$u_h = \hat{L}_N(h) \sum_k u_k(t) \mathrm{e}^{ikx} + c.c. \tag{12}$$

Using the same technique, S from (8) can be expressed as

$$S_N = \hat{L}_N(h)(-\partial_{xx})^{-1/2} \sum_k u_k(t) \mathrm{e}^{ikx} + c.c. \tag{13}$$

leading finally to

$$S_N(x,t) = \hat{L}_N(h)(-\partial_{xx})^{-1/2} \hat{L}_N^{-1}(h) u_h(x,t) \tag{14}$$

where $\hat{L}_N^{-1}$ is the operator inverse of $\hat{L}_N$.

C. Second order model

In the following we shall restrict ourselves to $N = 1$, resulting in a model of second order in the dependent variables $h(x,t),\ u_h(x,t)$. With

$$\hat{L}_1 = 1 + h\hat{D}, \qquad \hat{L}_1^{-1} = 1 - h\hat{D}\ ,$$

and the abbreviation$\hat{D} \equiv (-\partial_{xx})^{1/2}$ eq. (14) turns into

$$S_1 = \hat{D}^{-1}u_h - \hat{D}^{-1}(h\hat{D}u_h) + hu_h \tag{15}$$

where only terms up to the second order in u_h, h are included.

The complete model reads

$$\partial_t u_h = -\partial_x \left[\frac{1}{2}u_h^2 + h\right], \qquad \partial_t h = -\partial_x S_1 \tag{16}$$

with S_1 from (15)

Taking only linear terms into account ($\hat{L}_0 = 1$, $S_0 = \hat{D}^{-1}u_h$) one finds from (16)

$$\partial_{tt}h = \partial_x \hat{D}^{-1}\partial_x h = -\hat{D}h \tag{17}$$

where we have used the identity $\partial_{xx} = -(\hat{D})^2$. Eq. (17) possesses the well-known short-wave dispersion relation $\omega = \pm|k|^{1/2}$.

III. NUMERICAL METHOD AND RESULTS

A. Numerical method

To evaluate (15), the expressions

$$\hat{D}f(x), \qquad \hat{D}^{-1}f(x) \tag{18}$$

must be computed. The operator (11)

$$\hat{D}^n = (-\partial_{xx})^{n/2}$$

is nonlocal for odd n and (18) can be written as

$$\hat{D}^n f(x) = \int_0^\Gamma dx'\, G^{(n)}(x - x')f(x') \tag{19}$$

with the Green's function

$$G^{(n)}(x - x') = \frac{1}{\Gamma}\sum_{\ell=-\infty}^{\infty} |k_\ell|^n \mathrm{e}^{ik_\ell(x-x')}$$

and $k_\ell = 2\pi\ell/\Gamma$. However, evaluating (19) results in time-consuming computations of convolutions. Hence it is much more effective to compute (19) in Fourier space where it simply reads

$$\hat{D}^n \tilde{f}_\ell = |k_\ell|^n \tilde{f}_\ell \tag{20}$$

where $\tilde{f}_\ell$ is the discrete Fourier transform of $f(x_i)$ and can be obtained numerically by any standard Fast-Fourier transform (FFT) [5].

The derivatives with respect to x occurring in (16) are computed by centered differences to ensure the conservation of $\int_\Gamma u_h\, dx$, $\int_\Gamma h\, dx$. Finally, time integration is achieved by an explicit 2nd-order Adams-Bashforth algorithm [6].

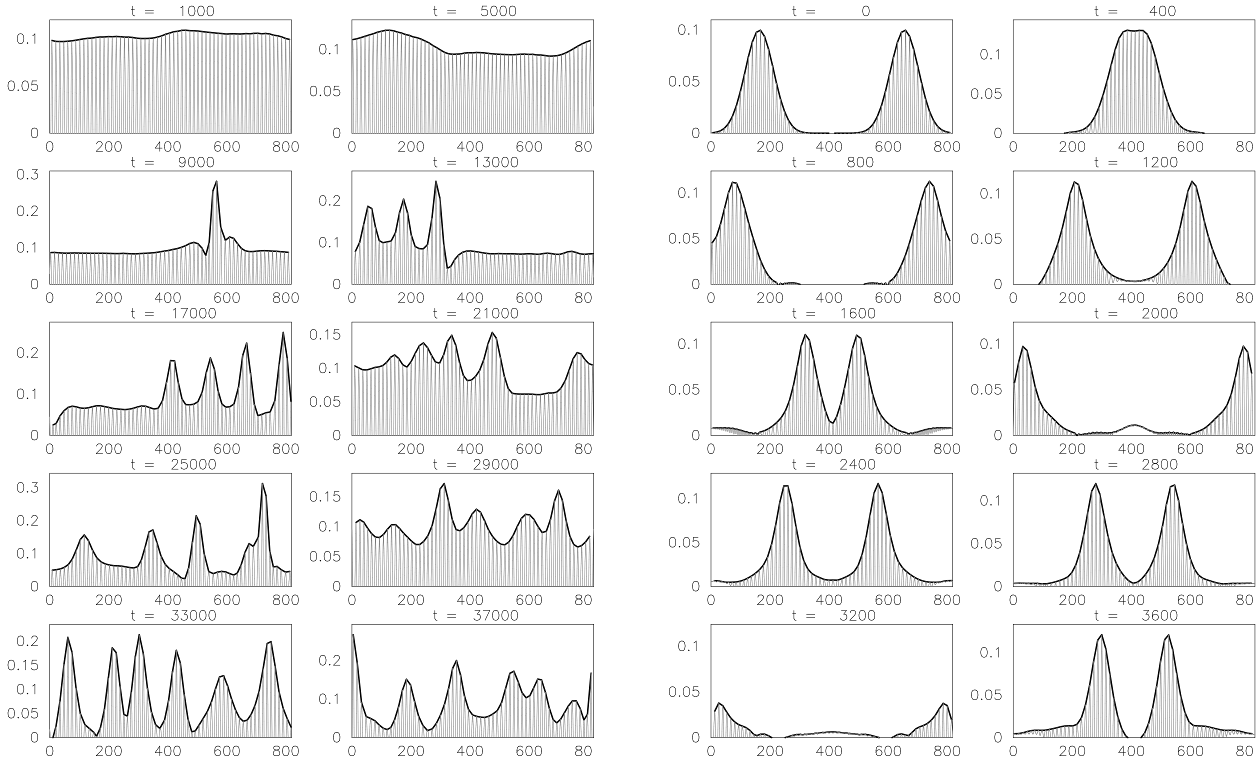

FIG. 1: Left frames: snapshots of a BF unstable wave train. Right frames: Two counter-propagating waves enveloped by Gaussians.

B. Results

The x-dimension is discretized with $N = 2048$ mesh points, the step sizes used are $\Delta x = 0.4$, $\Delta t = 0.003$, resulting in a side length of $\Gamma = 819.2$.

The time series in fig.1 (left frames) presents the evolution of a traveling wave to the right side with wave length $\lambda = 2\pi/k = \Gamma/60 \approx 13.7$ and amplitude $A = 0.1$. To trigger the side band (Benjamin-Feir) instability [3], a weak phase disturbance has been added. The speed of the (undisturbed) traveling wave follows as

$$c = \omega/k = 1/\sqrt{k} \approx 1.46,$$

giving a a Courant number $c\Delta t/\Delta x \approx 0.011$.

Another initial condition with two counter-traveling waves enveloped by Gaussians is used for the runs depicted in Fig.1, right frames. The Gaussian pulses travel in opposite direction and interact without changing their shapes strongly. Due to the periodic b.c. these interactions take place at equidistant time intervals.

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